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A Solution to the Navier-Stokes Equations Based Upon the Newton Kantorovich Method

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INTRODUCTION

The Newton-Kantorovich technique is a rigorous mathematical concept based on functional analysis that transforms a non-linear partial differential equation into a sequence of linear partial differential equations whose solutions converge to the solution of the original non-linear problem providing appropriate conditions can be satisfied. It has been employed operationally by Bellman[1] to solve a variety of non-linear problems with considerable success, and was more recently used by Morihara and Cheng[2] to produce a numerical solution of the steady-state Navier-Stokes equations in two-dimensions. Gabrielsen[3,4] has established a mathematical justification for the use of the technique in solving the non-steady Navier-Stokes equations in two dimensions.

The Newton-Kantorovich technique has major advantages over other techniques used to solve the non-steady Navier-Stokes equations. The main computational advantage is that a sequence of linear partial differential equations need only be solved rather than a single non-linear one. A linear partial differential equation can be solved numerically by simply solving a system of linear algebraic equations directly, whereas a non-linear equation requires an elaborate iteration procedure that may require more compute time and whose convergence is often difficult to achieve especially at high Reynolds numbers. A theoretical advantage of the technique is that explicit error estimates are attainable (that is, the difference between each linear solution and the solution to the original non-linear problem).

The obvious disadvantage of the technique is that more than one linear partial differential equation must be solved in order to achieve a solution. In fact, based on the results of previous investigations of the technique, the number of iterations of the procedure required for convergence of the technqiue would appear to be very problem dependent. Morihara and Cheng[2] applied the technique to a steady flow in the entrance region of a straight channel for various Reynolds numbers and found that the number of iterations required for convergence increased rapidly with Reynolds number. In addition, Gabrielsen[3] established an upper bound for the error associated with the technique as applied to the non-steady Navier-Stokes equations as a function of the number of successive iterations and the Reynolds number. His analysis indicated that the number of iterations required for convergence depended on the accuracy of the initial guess at the solution, and that an increasingly more accurate initial guess at the solution is required with increasing Reynolds number in order to achieve convergence of the procedure in the same number of iterations as required at lower Reynolds numbers. This Reynolds number dependence is due to the fact that the solution to the Navier-Stokes equations becomes progressively more non-uniform and more concentrated in a smaller region of the flow field with increasing Reynolds number. The convergence of the procedure must also depend on the shape of the geometry under consideration because the severity and location of these non-uniformities and concentrations will vary with geometry.

For expediency, the actual implementation of the procedure was accomplished by modifying an existing computer program written by Mehta[6]. Mehta's code is considered one of the most advanced computer programs in existence for numerically solving the laminar, unsteady, incompressible Navier-Stokes equations in two dimensions, in that the program is capable of determining the flow field about a flat plate, a circular or elliptical cylinder, and symmetric or cambered airfoils at arbitrary angles of attack including stall. The code is also capable of performing solutions at any Reynolds number for which the laminar flow assumption is reasonable. Another feature of the program that was particularly useful for the current study was that the equations of motion were transformed into the interior of the unit circle and the coordinate perpendicular to the surface of the geometric shape was stretched depending on the Reynolds number so that the solution produced by the transformed equation was relatively similar for all geometrics and Reynolds numbers. This feature also produces a transformed Navier-Stokes equation whose solution is relatively uniform throughout the region of calculation.

For the purposes of our investigation, Mehta's code was modified to solve each linear partial differential equation produced by the Newton-Kantorovich process. In addition, due to computer storage limitations, the program had to be simplified to perform computations for only symmetric geometries and flow fields. Test cases were performed for a circular cylinder at a Reynolds number of 15 and for a symmetrical

12% thick airfoil at zero angle of attack at Reynolds numbers of 10^3, 10^4, and 10^5. The results of the calculations using the Newton-Kantorovich procedure were in all cases compared with the results obtained for the same problem by the original unmodified version of the program.

MATHEMATICAL FORMULATION

The flow field exterior to the geometry under consideration is mapped into the unit circle. As a result of this transformation, the equation governing the unsteady, incompressible flow of a Newtonian fluid may be expressed in terms of the vorticity ω and the stream function Ψ as[7]

$$H^2 r^2 \frac{R}{L}\frac{\partial \omega}{\partial t} = \left(\frac{d\rho}{dr}\right)^2 r^2 \frac{\partial^2 \omega}{\partial \rho^2} + \left(\frac{d\rho}{dr} r + \frac{d^2\rho}{dr^2} r^2\right)\frac{\partial \omega}{\partial \rho} + \frac{\partial^2 \omega}{\partial \theta^2} - r\frac{d\rho}{dr}\frac{R}{L} J\left(\frac{\omega, \Psi}{\rho, \theta}\right) \qquad (1)$$

where

$$H^2 = \left(\frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r}\right) \Big/ r \qquad (2)$$

and the conservation-law form of the convective terms is expressed as

$$J\left(\frac{\omega, \Psi}{\rho, \theta}\right) = \frac{1}{3}\left[\frac{\partial}{\partial \rho}\left(\frac{\partial \Psi}{\partial \theta}\omega\right) + \frac{\partial \Psi}{\partial \theta}\frac{\partial \omega}{\partial \rho} + \frac{\partial}{\partial \theta}\left(\Psi \frac{\partial \omega}{\partial \rho}\right) - \frac{\partial}{\partial \theta}\left(\omega \frac{\partial \Psi}{\partial \rho}\right) - \frac{\partial \Psi}{\partial \rho}\frac{\partial \omega}{\partial \theta} - \frac{\partial}{\partial \rho}\left(\Psi \frac{\partial \omega}{\partial \theta}\right)\right] \qquad (3)$$

The disturbance stream function is defined as $\psi = \Psi - y$ where

$$r^2\left(\frac{d\rho}{dr}\right)^2 \frac{\partial^2 \psi}{\partial \rho^2} + \left(\frac{d\rho}{dr} r + \frac{d^2\rho}{dr^2} r^2\right)\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial \theta^2} = -H^2 r^2 \omega \qquad (4)$$

The mapping into the unit circle is accomplished by the transformation

$$z = \frac{1}{K} + \gamma + \frac{K c^2}{1 + \gamma K} \qquad (5)$$

where $z = x + iy$

$K = re^{i\theta}$

$\gamma = \xi + i\eta$

A proper choice of the constants γ and c invokes the solution for the flow over any one of a number of shapes including a flat plate, a circular or elliptical cylinder, or thick airfoils that may be symmetrical or cambered. The trailing edge of an airfoil shape may be rounded off by defining

$$c = \left[\xi + (1-\eta^2)^{1/2}\right](1-\delta) \quad \text{where } 0 < \delta \leq 1 \tag{6}$$

In order to minimize the effects of Reynolds number on the uniformity of the solution, the radial coordinate is stretched according to

$$\rho = (k_1 + k_2)^{-1}\left[\tanh^{-1}(r k_3 - k_4) + k_2\right] \tag{7}$$

with

$$k_3 = \left[\tanh k_1 - \tanh k_2\right]\left[1 - r_0/(r_0 - 1)\right]$$

and $$k_4 = \tanh k_2 - r_0\left[\tanh k_1 + \tanh k_2\right]/(r_0 - 1)$$

The constants r_o, k_1 and k_2 (all positive) determine the value of ρ. As r varies from r_o to 1, ρ varies from 0 to 1. The Reynolds number is $R = U\ell/\nu$ (ℓ and ν are, respectively, the chord and kinematic viscoscity) and L is the dimensionless chord.

The components of velocity u_1 and u_2 are defined in terms of the disturbance stream function as

$$u_1 = \frac{1}{rH}\left(\frac{\partial \psi}{\partial \theta} + \frac{\partial y}{\partial \theta}\right) \quad , \quad u_2 = -\frac{1}{H}\left(\frac{d\rho}{dr}\frac{\partial \psi}{\partial \rho} + \frac{\partial y}{\partial r}\right) \tag{8}$$

For Reynolds numbers much larger than unity, the vorticity in the flow field exists only near the body and in the wake. Away from this region, the flow is essentially irrotational. Therefore, the region of calculation is subdivided into two parts: a small viscous region and a large irrotational region bounded by ρ_o and ρ_r with $\theta_1 \leq \theta \leq \theta_2$ (see Figure 1).

The boundary conditions applied along the boundaries of the region of calculation are as follows:

1) On the surface ($\rho = 1$), the constraint of no slip is applied in the form

$$\Psi = 0 \quad (\text{or } \psi = -y) \tag{9}$$

and

$$\frac{\partial \Psi}{\partial \rho} = 0 \tag{10}$$

Condition (10) is used to calculate the surface vorticity from the stream function equation (4) with ψ replaced by Ψ.

2) Along the line of symmetry, the vorticity ω and the disturbance stream function ψ are specified to be zero.

3) The flow at the far boundary is constrained with first-order differential relations obtained from the Navier-Stokes equations by dropping the tangential derivative of the pressure and viscous terms, i.e., at the outer boundary, the inertia terms are dominant:

$$H r \frac{\partial \omega}{\partial t} = -\frac{d\rho}{dr} J\left(\frac{\omega, \Psi}{\rho, \theta}\right) \tag{11}$$

$$\frac{\partial \psi}{\partial \rho} = -\frac{dr}{d\rho}\left(u_2 H + \frac{\partial \eta}{\partial r}\right) \tag{12}$$

and u_2 is obtained from the θ component of the Navier-Stokes equation. That is,

$$\frac{\partial u_2}{\partial t} = -\left[\frac{1}{2rH}\frac{\partial(u_1^2+u_2^2)}{\partial \theta} + u_1\omega\right] \tag{13}$$

Note that, in (13), $\omega = 0$ when $\theta_1 \leq \theta \leq \theta_2$. At $t = 0$, the flow is irrotational (without circulation), i.e., $\omega = 0$ and $\psi = -yr^2$.

These boundary conditions are believed to be superior to specifying either potential flow or uniform velocity since eddies or vortices can pass through the downstream boundary. Also, since the velocity far away is not defined, the circulation there can change with time. Therefore, equation (11) correctly represents the vorticity transport through the downstream boundary. In equation (13), the absence of the tangential pressure derivative will not significantly affect the motion of a vortex through the boundary.

The surface pressure distribution is obtained by integrating the tangential component of the Navier-Stokes equation. That is,

$$p = \frac{L}{R}\frac{d\rho}{dr}\int_0^\theta \frac{\partial \omega}{\partial \rho}\, d\bar{\theta} \tag{14}$$

where $p(0) = 0$. The pressure coefficient C_p is, therefore, equal to $2p$. On the surface, the tangential stress is given by $\sigma_{12} = (L/R)\omega$. Both p and σ_{12} are made dimensionless with ρU^2. The coefficients of

lift, drag, and moment around the origin of the z plane (defined as positive in the counter clockwise direction) are given by

$$C_L = \frac{\text{lift}}{\frac{1}{2}\rho U^2 L} = C_{LP} + C_{LF} = -\frac{2}{L}\int_0^{2\pi} p\,\frac{\partial x}{\partial \theta}\,d\theta - \frac{2}{R}\int_0^{2\pi} \omega\,\frac{\partial y}{\partial \theta}\,d\theta \tag{15}$$

$$C_D = \frac{\text{drag}}{\frac{1}{2}\rho U^2 L} = C_{DP} + C_{DF} = \frac{2}{L}\int_0^{2\pi} p\,\frac{\partial y}{\partial \theta}\,d\theta - \frac{2}{R}\int_0^{2\pi} \omega\,\frac{\partial x}{\partial \theta}\,d\theta \tag{16}$$

$$C_M = \frac{\text{moment}}{\frac{1}{2}\rho U^2 L^2} = C_{MP} + C_{MF} = -\frac{2}{L^2}\int_0^{2\pi} p\left(x\,\frac{\partial x}{\partial \theta} + y\,\frac{\partial y}{\partial \theta}\right)d\theta - \frac{2}{R}\int_0^{2\pi} \omega\left(-y\,\frac{\partial x}{\partial \theta} + x\,\frac{\partial y}{\partial \theta}\right)d\theta \tag{17}$$

where the subscripts P and F, respectively, represent the contributions due to pressure and viscous forces.

The Newton-Kantorovich procedure transforms equation (1) into a sequence of linear partial differential equations (see Appendix A for development) each of which has the form,

$$H^2 r^2 \frac{R}{L}\frac{\partial \omega_{m+1}}{\partial t} = \left(\frac{d\rho}{dr}\right)^2 r^2 \frac{\partial^2 \omega_{m+1}}{\partial \rho^2} + \left(\frac{d\rho}{dr} r + \frac{d^2\rho}{dr^2} r^2\right)\frac{\partial \omega_{m+1}}{\partial \rho} + \frac{\partial^2 \omega_{m+1}}{\partial \theta^2}$$
$$- r\frac{d\rho}{dr}\frac{R}{L}\left\{J\left(\frac{\omega_m, \Psi_{m+1}}{\rho, \theta}\right) + J\left(\frac{\omega_{m+1}, \Psi_m}{\rho, \theta}\right) - J\left(\frac{\omega_m, \Psi_m}{\rho, \theta}\right)\right\} \tag{18}$$

where $$\nabla^2 \psi = -\omega \tag{19}$$

Since the non-linearity in equation (1) occurs only in the Jacobian term, the Newton-Kantorovich transformation affects this term alone. The subscript m+1 refers to the variables that must be determined by solving the current linear partial differential equation and the subscript m refers to the variables that were determined by solving the

previous linear partial differential equation in the sequence. The Jacobian terms on the right hand side of equation (18) are now linear because only the variables with subscript m+1 are unknown. A sequence of these equations are solved such that the solutions ψ_n and ω_n converge to the exact solutions ψ and ω of the non-linear problem described by equations (1) and (4). As shown in references 1 and 3, the successive approximations described by equation (18) converge quadratically.

NUMERICAL FORMULATION

Two computer programs were used for performing the calculations of the current study. The first computer program utilized the same numerical technique as that used by Mehta[6] for solving the non-steady incompressible Navier-Stokes equations for arbitrary airfoils at angle of attack but was a simpler version only applicable to symmetrical airfoils at zero angle of attack. This computer program is referred to as the original or unmodified program and was used as a means of checking the validity of the Newton-Kantorovich solutions. The second program was similar to the first one except that it was modified so as to solve the sequence of linear partial differential equations required by the Newton-Kantorovich concept and expressed in equation (18). The details of the numerical procedure used for the original program may be found in Reference 6. The numerical procedure used in utilizing the Newton-Kantorovich concept is shown here.

Each Newton-Kantorovich linear system was solved by a three-point backward time differencing and implicit factored central space differencing scheme.

The spatially factored, time-differenced, expression for the Newton-Kantorovich version of the vorticity transport equation (18) is

$$\left(1-\frac{2\Delta t}{TA}\delta_\theta\right)\left(1-\frac{2\Delta t}{TA}\delta_\rho\right)\omega_{m+1}^{n} = \frac{2\Delta t}{TA}\left(Q+F+G\right)-\frac{1}{T}\left(T_1\omega_{m+1}^{n-1}+T_2\omega_{m+1}^{n-2}\right)$$

$$+\frac{4(\Delta t)^2}{T^2}\frac{\delta_\theta}{A}\frac{\delta_\rho}{A}\omega_{m+1}^{n} \qquad (20)$$

where the operators δ_θ and δ_ρ are given as

$$\delta_\theta = \frac{\partial^2}{\partial\theta^2} + \frac{r}{3}\frac{d\rho}{dr}\frac{R}{L}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\Psi^n_{m+1}}{\partial\rho}\right) + \frac{\partial\Psi_{m+1}}{\partial\rho}\frac{\partial}{\partial\theta}\right] \tag{21}$$

$$\delta_\rho = \left(\frac{d\rho}{dr}\right)^2 r^2\frac{\partial^2}{\partial\rho^2} + \left(\frac{d\rho}{dr}r + \frac{d^2\rho}{dr^2}r^2\right)\frac{\partial}{\partial\rho} - \frac{r}{3}\frac{d\rho}{dr}\frac{R}{L}\left[\frac{\partial}{\partial\rho}\left(\frac{\partial\Psi_{m+1}}{\partial\theta}\right)^n + \frac{\partial\Psi^n_{m+1}}{\partial\theta}\frac{\partial}{\partial\rho}\right] \tag{22}$$

and the cross derivative terms Q, F, and G as

$$Q = -\frac{r}{3}\frac{d\rho}{dr}\frac{R}{L}\left[\frac{\partial}{\partial\theta}\left(\Psi^n_m\frac{\partial\omega^n_{m+1}}{\partial\rho}\right) - \frac{\partial}{\partial\rho}\left(\Psi^n_m\frac{\partial\omega^n_{m+1}}{\partial\theta}\right)\right] \tag{23}$$

$$F = -\frac{1}{3}\frac{d\rho}{dr}\frac{R}{L}\left[\frac{\partial}{\partial\rho}\left(\frac{\partial\Psi^n_{m+1}}{\partial\theta}\omega^n_m - \frac{\partial\Psi^n_m}{\partial\theta}\omega^n_m\right) + \frac{\partial\Psi^n_{m+1}}{\partial\theta}\frac{\partial\omega^n_m}{\partial\rho} - \frac{\partial\Psi^n_m}{\partial\theta}\frac{\partial\omega^n_m}{\partial\rho}\right.$$
$$\left. + \frac{\partial}{\partial\theta}\left(\Psi^n_{m+1}\frac{\partial\omega^n_m}{\partial\rho} - \Psi^n_m\frac{\partial\omega^n_m}{\partial\rho}\right)\right] \tag{24}$$

$$G = \frac{r}{3}\frac{d\rho}{dr}\frac{R}{L}\left[\frac{\partial}{\partial\theta}\left(\frac{\partial\Psi^n_{m+1}}{\partial\rho}\omega^n_m - \frac{\partial\Psi^n_m}{\partial\rho}\omega^n_m\right) + \frac{\partial\Psi^n_{m+1}}{\partial\rho}\frac{\partial\omega^n_m}{\partial\theta} - \frac{\partial\Psi^n_m}{\partial\rho}\frac{\partial\omega^n_m}{\partial\theta}\right.$$
$$\left. + \frac{\partial}{\partial\rho}\left(\Psi^n_{m+1}\frac{\partial\omega^n_m}{\partial\theta} - \Psi^n_m\frac{\partial\omega^n_m}{\partial\theta}\right)\right] \tag{25}$$

The values of the three-point backward time difference parameters are

$$T = 3$$

$$T_1 = -4$$

$$T_2 = 1$$

and the tranformation parameter A is

$$A = H^2 r^2 \frac{R}{L}$$

Consistent with the spatial factoring concept, equation (20) is split into two equations whose finite difference analogies each produce tri-diagonal systems of equations. That is,

$$\left(1 - \frac{2\Delta t}{TA}\delta_\rho\right)\omega^n_{m+1} = \omega^*_{m+1} \tag{26}$$

and

$$\left(1 - \frac{2\Delta t}{TA}\delta_\theta\right)\omega^*_{m+1} = \frac{2\Delta t}{TA}\left(Q+F+G\right) - \frac{1}{T}\left(T_1\omega^{n-1}_{m+1} + T_2\omega^{n-2}_{m+1}\right)$$

$$+ \frac{4(\Delta t)^2}{T^2}\frac{\delta_\theta}{A}\frac{\delta_\rho}{A}\omega^n_{m+1} \tag{27}$$

The spatial derivatives associated with equations (26) and (27) as well as equation (19) were approximated by central differencing formulas everywhere except at the boundaries of the flow field. The truncation error for the vorticity equation is $O[(\Delta\rho)^2 + (\Delta\theta)^2 + (\Delta t)^2]$ except at the first two time steps where the temporal error is $O[\Delta t]$ because a two-point backward difference formula is required such that $T_1 = 2$, $T_2 = -2$, and $T_3 = 0$. The truncation error for the disturbance stream function equation is $O[(\Delta\rho)^2 + (\Delta\theta)^2]$.

The no-slip boundary condition represented by equations (9) and (10) is reformulated in terms of vorticity. The finite difference expression for this boundary condition is

$$\omega_{JL_{m+1}} = \frac{1}{(2+3\Delta r)H^2}\left[\frac{6\left(\psi_{JL_{m+1}} - \psi_{JL-1_{m+1}}\right)}{\Delta r^2} - \left(H^2r^2\omega\right)_{JL-1_{m+1}} + \frac{6}{\Delta r}\frac{\partial\psi}{\partial r}\right.$$

$$\left. -3\frac{\partial^2\psi}{\partial r^2} + \Delta r\frac{\partial^3\psi}{\partial r^3}\right] \tag{28}$$

The truncation error for this formulation is $O[\Delta r^2]$. In addition, the condition $\Psi = 0$ is also satisfied on this boundary. Along the lines of symmetry the boundary conditions $\omega = 0$ and $\psi = 0$ are specified. At the outer boundary, the finite difference forms of equations (26) and (27) are used to describe the vorticity condition except that the diffusion terms are deleted and forward space differencing is used in the ρ direction. This formulation results in a first order truncation error in the space variable ρ. The disturbance stream function at this boundary is obtained from the finite-difference form of equations (12) and (13):

$$\psi^n_{i,1_{m+1}} = \frac{1}{3}\left[\frac{2\Delta\rho}{(d\rho/dr)_1}\left(Hu_2^n + \frac{\partial\eta}{\partial r}\right)_{i,1_{m+1}} + 4\psi^n_{i,2_{m+1}} - \psi^n_{i,3_{m+1}}\right] \tag{29}$$

where

$$\left(u_2\right)^n_{i,1_{m+1}} = \frac{1}{T_1}\left[-2\Delta t\left(u_1^n\omega^n\right)_{i,1_{m+1}} + \frac{(u_1^2+u_2^2)^n_{i+1,1_{m+1}} - (u_1^2+u_2^2)^n_{i-1,1_{m+1}}}{4\Delta\theta(rH)_{i,1}} - \left(T_2u_2^{n-1} + T_3u_2^{n-2}\right)_{i,1_{m+1}}\right] \tag{30}$$

The present values of the vorticity at a wall grid point is determined from

$$\omega^k_{m+1} = \omega^{k-1}_{m+1} + \beta_1\left(\Delta\omega\right)^k_{m+1} \tag{31}$$

where β_1 is a relaxation parameter and k is an iteration counter.

Once the solution to the linear system of equations has been obtained as described above at a particular time step, the Newton-Kantorovich procedure is invoked to produce a new linear system and the process is repeated as many times as required for convergence to the solution of the non-linear system for that time step. Then the entire procedure is advanced to the next time step. The initial guess, ω_o, at the solution for each time step, was determined on the basis of an extrapolation of the converged solutions at the two previous time steps. That is,

$$\omega_o^{n+1} = 2\,\omega^n - \omega^{n-1} \tag{32}$$

This approximation has a truncation error of $O(\Delta t^2)$. At the first time step ω_o was set to zero and the potential flow stream function was prescribed for ψ_o. At the second time step ω_o and ψ_o are set equal to the converged solutions at the first time step. For subsequent time steps, ψ_o was set equal to the converged solution at the previous time step.

Since the truncation error of the Newton-Kantorovich approximation is $O[\Delta\omega \cdot \Delta\psi]$, the truncation error of the initial guess at the first time step is not determinable in terms of Δt, but at the second time step it is $O[\Delta t^3]$ and at subsequent time steps it is $O[\Delta t^4]$.

Computations of pressure coefficients on the surface and a determination of the loads are made for each time t with a finite-difference integration formula derived by combining two four-point expressions

(in order to have a lower effective truncation error). The integral of any function f (ρ) between the modal points i and i + 1 on the surface are expressed as

$$I = \frac{1}{30}\left(11 I_1 + 19 I_2\right)$$

where $I_1 = \frac{1}{24}\Delta\theta\left(9f_i + 19f_{i+1} - 5f_{i+2} + f_{i+3}\right)$

$$I_2 = \frac{1}{24}\Delta\theta\left(-f_{i-1} + 13f_i + 13f_{i+1} - f_{i+2}\right)$$

This formula is used first to determine the pressure from equation (14) and then is used to compute the loads. The pressure calculations require the normal vorticity gradient, which is represented by

$$\frac{\partial\omega}{\partial\rho} = \left(\omega_{JL_{m+1}} - \omega_{JL-1_{m+1}}\right)\frac{1}{\Delta\rho}$$

This equation has a truncation error of 0 (Δρ). A formula with a smaller truncation error is not used because the derivation of the surface vorticity with a truncation error of $0(\Delta\rho)^2$ requires the use of a formula identical to the above equation for $\partial(H^2 r^2\omega)/\partial\rho$.

DISCUSSION OF NUMERICAL RESULTS

The two computer programs discussed in the previous section were both used to determine the flow field about a circular cylinder at a Reynolds number of 15, and a 12% thick symmetrical airfoil at zero angle of attack at Reynolds numbers of 10^3, 10^4, and 10^5. In all cases, the bodies under consideration were impulsively started. The cases for the circular cylinder at a Reynolds number of 15 and the airfoil at a Reynolds number of 10,000 were computed until a steady state was achieved, while the other cases were run for early time only. The results calculated by both programs were compared with each other as a basis for determining the number of iterations of the Newton-Kantorovich procedure required for convergence. Figures 2 through 5 illustrate some of the numerical results and table I summarizes the convergence characteristics of the technique for these calculations. In general, the results obtained by both techniques were in excellent agreement with each other. As seen in the figures, the maximum differences between the two sets of results was about 1%. This difference did not decrease with an increase in the number of Newton-Kantorovich iterations but did decrease with a finer spatial grid. This indicates that the difference between the solutions was due to the slightly different finite difference approximations to the Navier-Stokes equations produced by the two techniques.

In addition, Table I shows that finer spatial grids were required at the higher Reynolds numbers in order to achieve the same level of agreement. This was due to the fact that the solutions became more non-uniform in the circumferential direction with increasing Reynolds number. Since the circumferential coordinate was not stretched with Reynolds number as was the radial coordinate, the number of grid elements in the circumferential direction had to be increased in order to maintain the same level of accuracy in the regions of large gradients in the solution.

The convergence characteristics of the Newton-Kantorovich procedure were excellent for all test cases performed. In fact, after the first several time steps of each test case, the initial guesses for succeeding time steps were sufficiently accurate that only a single iteration was required to achieve convergence; and thus the technique became completely non-iterative with respect to the non-linearity in the equations. This is illustrated in Figure 6 for the Reynolds number 10^4 where the vorticity at a particular point in the flow field is plotted versus time using one Newton-Kantorovich iteration and is compared to the same case using two iterations. For the first several time steps, two iterations were required for convergence of the Reynolds number 15, 10^3, and 10^4 cases, and a third iteration was required for convergence of the Reynolds number 10^5 case. These succeeding iterations at the earliest time steps were required because the initial guesses at the solution for early time were very inaccurate (see Numerical Formulation).

Smaller time steps were required at the higher Reynolds numbers to achieve a sufficiently accurate initial guess at the solution to preserve the number of iterations required for convergence of the procedure. However, smaller time steps were required anyway to achieve convergence of the surface vorticity boundary condition at these Reynolds numbers. Hence, these time step restrictions are consistent with the solution procedure and not a limitation due to the Newton-Kantorovich process.

CONCLUSIONS

Based on the results obtained in the current study, the Newton-Kantorovich technique can be successfully applied to the numerical solution of the laminar, non-steady, incompressible, two-dimensional Navier-Stokes equations. The convergence characteristics of the technique were excellent for all geometries and Reynolds numbers tested. In fact, except for the first several time steps, the procedure requires only one iteration to achieve suitable convergence; that is, the procedure becomes non-iterative with regard to the handling of the non-linear terms in the Navier-Stokes equations. This conclusion indicates a potential for significant reduction in computation time over other current iterative techniques.

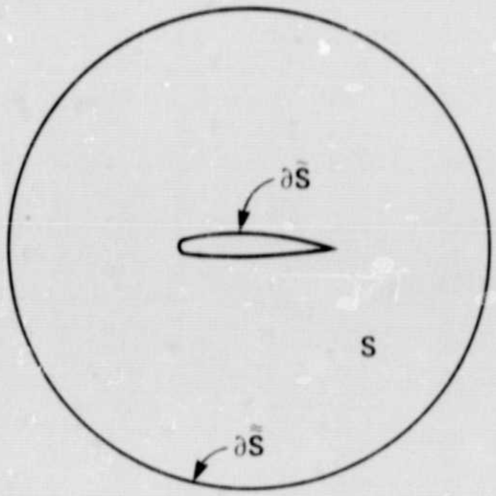
APPENDIX

The nonlinear, nonstationery Navier-Stokes equations, which can be described mathematically by

$$\Delta\psi_t - \nu\Delta\Delta\psi = \psi_y\Delta\psi_x - \psi_x\Delta\psi_y \quad \text{in } S \tag{1}$$

$$\psi(\partial\tilde{S}) = \tilde{b}_1,\ \psi(\partial\tilde{\tilde{S}}) = \tilde{\tilde{b}}_1,$$

$$\Delta\psi(\partial\tilde{S}) = \tilde{b}_2,\ \Delta\psi(\partial\tilde{\tilde{S}}) = \tilde{\tilde{b}}_2$$



with $\psi(S)\big|_{t=0} = \Phi(S)$

can, equivalently, be described in the form

$$P(\psi) = 0 \quad \text{in } S \tag{2}$$

$$\psi(\partial S) = b_1,\ \Delta\psi(\partial S) = b_2,\ \psi(S)\big|_{t=0} = \Phi(S).$$

Operator P of (2) will be considered a mapping from the Banach Space $C_1^{4}(S)$ into the Banach space $C_o^{o}(S)$, where $C_T^{N}(S) \equiv \{f(x,y,t)\,|$ N times continuously differentiable in $(x,y)\in S$,T times continuously differentiable in

$t, 0 \leq t < \infty$, inclusive of spatial derivatives up through order $[N/2]\}$. Note that $[\,]$ denotes greatest integer part. $C_T^N(S)$ will be considered under the norm

$$\|\psi\|_{C_T^N} = \sum_{i=0}^{T} \sum_{n=0}^{[\frac{N}{2}]} \sum_{m=0}^{n} \max_{x,y\in S} \left| \frac{\partial^{n+i}\psi}{\partial t^i \partial x^m \partial y^{n-m}} \right| + \sum_{n=[\frac{N}{2}]+1}^{N} \sum_{m=0}^{n} \max_{\substack{x,y\in S\\ 0\le t<\infty}} \left| \frac{\partial^n \psi}{\partial x^m \partial y^{n-m}} \right|$$

In order to apply Newton's method as generalized to function spaces by Kantorovich[5], it is necessary to determine whether or not P', the Frechet derivative of operator P of equation (2), P", and $(P')^{-1}$ exist, and if they exist, upper bounds for $\|P\|$, $\|P''\|$, and $\|(P')^{-1}\|$ must be determined.

It can be readily shown (see (3),(4)) that $P'(\psi_0)$ exists for arbitrary $C_1^4(S)$, and that

$$P'(\psi_0) = \Delta\frac{\partial}{\partial t} - \nu\Delta\Delta - \psi_{0_y}\Delta\frac{\partial}{\partial x} - \psi_{0_x}\frac{\partial}{\partial y} + \psi_{0_x}\Delta\frac{\partial}{\partial y} + \Delta\psi_{0_y}\frac{\partial}{\partial x}$$

Similarly, it can be shown ((3)(4)) that

$$P''(\psi_0)\phi\theta = \phi_y\Delta\theta_x + \theta_y\Delta\phi_x - \theta_x\Delta\phi_y - \phi_x\Delta\theta_y$$

for arbitrary ψ_0, ϕ, θ in $C_1^4(S)$. Also,

If $$\|P(\psi_0)\|\,\|P'(\psi_0)^{-1}\|^2\,\|P''(\psi_0)\| < \frac{1}{2}$$

for some initial guess $\psi_0 \in C_1^4(S)$, then the algorithm

$$P(\psi_m) + P'(\psi_m)(\psi_{m+1} - \psi_m) = 0,\ \psi_m(\partial S) = b_1,\ \Delta\psi_m(\partial S) = b_2,\ \psi_m\Big|_{t=0} = \Phi(S) \quad (3)$$

converges to the solution of (2) (see (5)).

Therefore, it remains to determine if and when $P'(\psi_o)^{-1}$ exists and to obtain an upper estimate for $\| P'(\psi_0)^{-1} \|$.

To this end, let m = 0 in Eq. (3). Therefore,

$$0 = P(\psi_0) + P'(\psi_0)(\psi_1 - \psi_0) \quad \text{in } C_0^0$$

$$\psi_0(\partial S) = b_1,\ \psi_1(\partial S) = b_1,\ \Delta\psi_0(\partial S) = b_2,\ \Delta\psi_1(\partial S) = b_2$$

$$\psi_0\Big|_{t=0} = \Phi(S),\ \text{and}\ \psi_1\Big|_{t=0} = \Phi(S).$$

Let $\overline{\Psi} = \psi_1 - \psi_0$; therefore, $-P(\psi_0) = P'(\psi_0)\overline{\Psi}$. If $\overline{\Psi}$ exists, then

$$\overline{\Psi} = -P'(\psi_0)^{-1} P(\psi_0).$$

Hence, we seek a solution $\overline{\Psi}$ of the problem

$$0 = P(\psi_0) + P'(\psi_0)\overline{\Psi},\ \overline{\Psi}(\partial S) = 0,\ \Delta\overline{\Psi}(\partial S) = 0, \text{and } \overline{\Psi}\Big|_{t=0} = 0. \quad (4)$$

Let $P'(\psi_0) = \Delta\frac{\partial}{\partial t} - \nu\Delta\Delta + A$,
where A is a linear operator $C_0^3(S)$ into $C_0^0(S)$.

$$A(\psi_0)\phi = -\psi_{0_y}\Delta\phi_x - \Delta\psi_{0_x}\phi_y + \psi_{0_x}\Delta\phi_y + \Delta\psi_{0_y}\phi_x.$$

Equation (4) can, therefore, be expressed by

$$\Delta\overline{\Psi}_t - \nu\Delta\Delta\overline{\Psi} = -P(\psi_0) - A(\psi_0)\overline{\Psi},\ \overline{\Psi}(\partial S) = 0,\ \Delta\overline{\Psi}(\partial S) = 0,\ \overline{\Psi}\Big|_{t=0} = 0 \quad (5)$$

Let $\bar{\bar{\Psi}}$ be defined as follows:

$$\bar{\bar{\Psi}}(x,y,t) = -\int_S\int_0^t\int_S \left\{[A(\psi_0)\bar{\bar{\Psi}}](x'',y'',\tau) + P(\psi_0)(x'',y'',\tau)\right\} G^\nu(x,y,x',y',x'',y'',t-\tau) \cdot dx''dy''d\tau dx'dy';$$

where

$$G^\nu(x,y,x',y',x'',y'',t) = G(x,y,x',y')H^\nu(x',y';x'',y'',t)$$

It can be directly shown that if $\bar{\bar{\Psi}}$ of (6) exists, then $\bar{\bar{\Psi}}$ satisfies (4), where G is the Green's Function of La Place's equation for S and H^ν the kernel function for the heat equation

$$\phi_t - \nu\Delta\phi = 0 \quad \text{in } S$$

$$\phi(\partial S) = 0, \quad \phi\Big|_{t=0} = 0$$

In order to determine sufficient conditions under which $\bar{\bar{\psi}}$ of equation (6) will exist, let B denote the linear operation:

$$Bq = -\int_S\int_0^t\int_S q(x'',y'',\tau)G^\nu(x,y,x',y',x'',y'',t-\tau)\,dx'dy'd\tau dx''dy''.$$

Equation (6) therefore takes the form

$$(I - BA)\bar{\bar{\Psi}} = BP(\psi_0)$$

Let $C_0^3(S) \equiv$ the completion of $C_0^3(S)$ under the norm

$$\|\psi\| = \max_{\substack{x,y\in S\\0\le t<\infty}}|\psi| + \max_{\substack{x,y\in S\\0\le t<\infty}}|\psi_x| + \max_{\substack{x,y\in S\\0\le t<\infty}}|\psi_y|$$

$$+ \max_{\substack{x,y\in S\\0\le t<\infty}}|\psi_{xx}| + \max_{\substack{x,y\in S\\0\le t<\infty}}|\psi_{xy}| + \max_{\substack{x,y\in S\\0\le t<\infty}}|\psi_{yy}|$$

$$+ \max_{\substack{x,y\in S\\0\le t<\infty}}|\Delta\psi_x| + \max_{\substack{x,y\in S\\0\le t<\infty}}|\Delta\psi_y| \, .$$

If (BA) is considered a mapping from $\tilde{C}_0^{\,3}(S)$ into $\tilde{C}_0^{\,3}(S)$, and $\|BA\| < 1$, then $(I - BA)^{-1}$ exists, and $\bar{\bar{\psi}} = (I - BA)^{-1} BP(\psi_0)$. Therefore, in order to show that $\bar{\bar{\psi}}$ exists, it is sufficient to show that $\|B\|\,\|A\| < 1$. To this end consider A as a mapping from $\tilde{C}_0^{\,3}$ into $C_0^{\,0}$. For $\psi \in C_0^0$,

$$\|\psi\|_{C_0^0} = \max_{\substack{x,y\in S\\0\le t<\infty}}|\psi| \; ; \quad \text{for } \psi \in \tilde{C}_0^3$$

$$\|\psi\|_{\tilde{C}_0^3} = \max_{\substack{x,y\in S\\0\le t<\infty}}\left(|\psi| + |\psi_x| + |\psi_y| + |\psi_{xx}| + |\psi_{yy}| + |\psi_{xy}| + |\Delta\psi_x| + |\Delta\psi_y|\right).$$

Since $\|A\| = \sup\limits_{\|\phi\|_{\tilde{C}_0^3}=1} \|A\phi\|$ by definition,

$$A(\psi_0)\,\phi = -\psi_{0_y}\,\Delta\phi_x - \Delta\psi_{0_x}\,\phi_y + \psi_{0_x}\,\Delta\phi_y + \Delta\psi_{0_y}\,\phi_x \, ,$$

and

$$\|A\phi\|_{C_0^o} = \max |A\phi| = \max |-\psi_{0_y}\Delta\phi_x - \Delta\psi_{0_x}\phi_y + \psi_{0_x}\Delta\phi_y + \Delta\psi_{0_y}\phi_x|,$$

it follows that $\|A\| \le M_{\psi_0}$, where

$$M_{\psi_0} = \max_{\substack{x,y \in S \\ 0 \le t < \infty}} (|\psi_{0_y}|, |\psi_{0_x}|, |\Delta\psi_{0_x}|, |\Delta\psi_{0_y}|).$$

Consider B as a mapping from C_o^o into $\tilde{C}_0^3$. $\|B\| = \sup_{\|z\|_{C_0^o}=1} (\|Bz\|_{\tilde{C}_0^3})$,

by definition. Therefore, since

$$\|Bz\|_{C_0^3} = \max\Big(|Bz| + |(Bz)_x| + |(Bz)_y| + |(Bz)_{xx}| + |(Bz)_{xy}| + |(Bz)_{yy}| + |(\Delta Bz)_x| + |(\Delta Bz)_y|\Big),$$

$$\|B\| \le \max_{\substack{x,y \in S \\ 0 \le t < \infty}} \int_S \int_0^t \int_S H^\nu(x',y',x'',y'',t-\tau)[G(x,y,x',y') + |G_y| + |G_x| + |G_{xx}| + |G_{yy}| + |G_{xy}|]\,dx''dy''d\tau dx'dy'$$

$$+ \max \int_0^t \int_S [|H_x^\nu(x,y,x'',y'',t-\tau)| + |H_y^\nu(x,y,x'',y'',t-\tau)|]\,dx''dy''d\tau.$$

Let

$$N_B = \max \int_S \int_0^t \int_S H^\nu(x',y',x'',y'',t-\tau)(G + |G_x| + |G_y| + |G_{xx}| + |G_{yy}| + |G_{xy}|)\,dx''dy''d\tau,$$

and

$$M_{\psi_0} = \max(|\psi_{0_y}|, |\psi_{0_x}|, |\Delta\psi_{0_x}|, |\Delta\psi_{0_y}|).$$

Therefore, for $M_{\psi_0} N_B < 1$, $\bar{\bar{\psi}}$ exists, and $\bar{\bar{\psi}} = (I - BA)^{-1} BP(\psi_0)$. Hence, it follows directly that $-P'(\psi_0)^{-1} = (I - BA)^{-1} B$ and that

$$\| P'(\psi_0)^{-1} \| \leq \frac{\|B\|}{1 - \|B\| \|A\|} \leq \frac{N_B}{1 - N_B M_{\psi_0}} .$$

From the preceding, it directly follows that if ψ_0, the initial educated guess, is judiciously chosen, a sequence of functions $\psi_n (n = 1, 2, \ldots)$ can be directly constructed that conver[illegible] the exact solution of the nonlinear, nonstationary Navier-Stokes problem as expressed by Equation (1). In addition, an explicit error estimate can be directly determined for each approximate solution ψ_n.

Explicitly then, for

$$\| P(\psi_0) \| < \frac{(1 - N_B M_{\psi_0})^2}{2 N_B^2}$$

with $M_{\psi_0} N_B < 1$, the sequence of ψ_n's determined by Equation (3) exists and converges to the exact solution ψ^* of Equation (1).

Moreover, under these conditions, the method yields the explicit error estimate

$$\| \psi^* - \psi_n \| \leq 2^{2^n - n} \left(\frac{N_B}{1 - M_{\psi_0} N_B} \right)^{2^{n+1} - 1} \| P(\psi_0) \|^{2^n} .$$

The actual application of the Newton-Kantorovich method to equation (1) produces a sequence of equations of the following form:

$$\frac{\partial}{\partial t}(\Delta\psi_{m+1}) - \nu\Delta\Delta\psi_{m+1} = \psi_{m_y}\Delta\psi_{m+1_x} + \Delta\psi_{m_x}\psi_{m+1_y} - \psi_{m_x}\Delta\psi_{m+1_y} - \Delta\psi_{m_y}\psi_{m+1_x}$$

$$- \Delta\psi_{m_x}\psi_{m_y} + \Delta\psi_{m_y}\psi_{m_x}$$

Define the vorticity, ω, as

$$\omega = -\Delta\psi$$

Hence, the final form of the equation becomes

$$\frac{\partial\omega_{m+1}}{\partial t} - \nu\Delta\omega_{m+1} = -J\left(\frac{\omega_{m+1},\psi_m}{x,y}\right) - J\left(\frac{\omega_m,\psi_{m+1}}{x,y}\right) + J\left(\frac{\omega_m,\psi_m}{x,y}\right)$$

Transforming this equation into the interior of the unit circle and transforming the radial coordinate to a new variable ρ that allows for stretching of the solution in that direction produces

$$H^2 r^2 \frac{\partial\omega_{m+1}}{\partial t} = \left(\frac{d\rho}{dr}\right)^2 r^2 \frac{\partial^2\omega_{m+1}}{\partial\rho^2} + \left(\frac{d\rho}{dr} r + \frac{d^2\rho}{dr^2} r^2\right)\frac{\partial\omega_{m+1}}{\partial\rho} + \frac{\partial^2\omega_{m+1}}{\partial\theta^2}$$

$$- r\frac{d\rho}{dr}\frac{R}{L}\left\{J\left(\frac{\omega_m,\Psi_{m+1}}{\rho,\theta}\right) + J\left(\frac{\omega_{m+1},\Psi_m}{\rho,\theta}\right) - J\left(\frac{\omega_m,\Psi_m}{\rho,\theta}\right)\right\}$$

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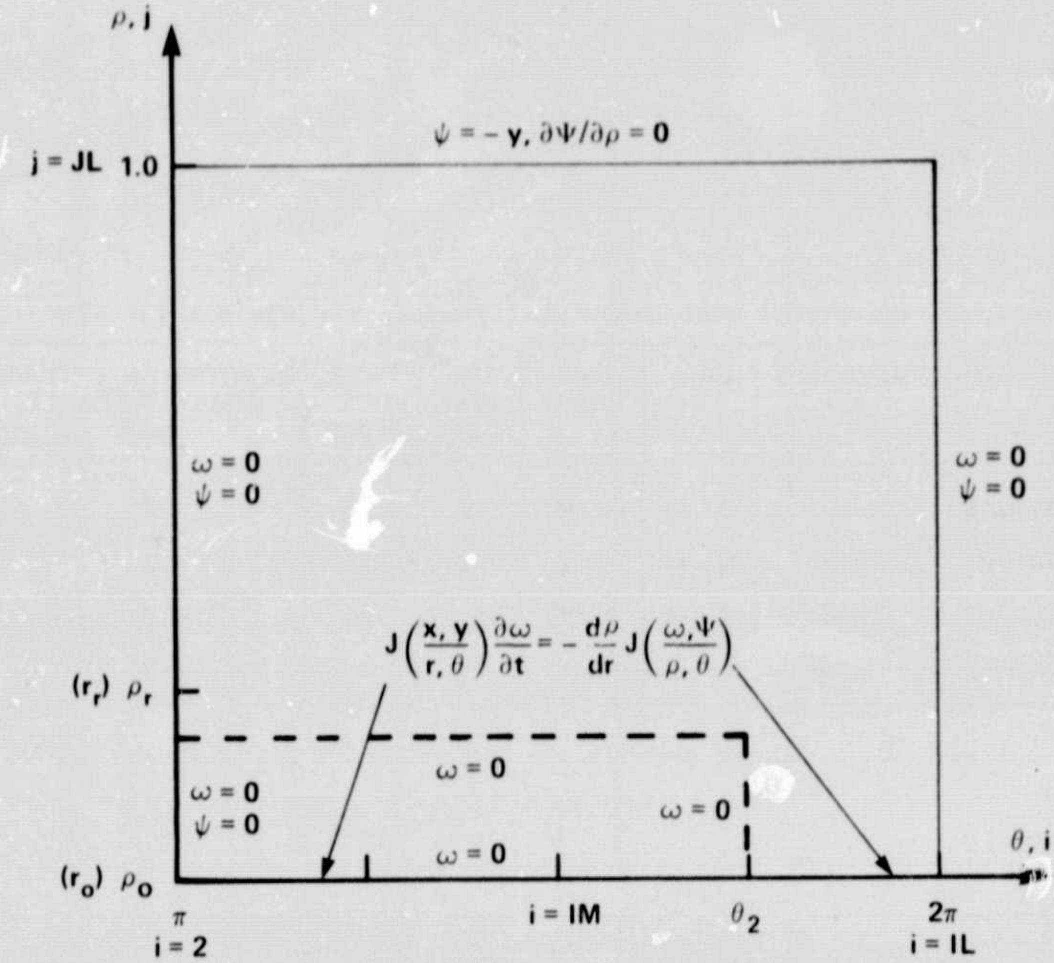
Table 1 Summary of test cases performed during the current study.

GEOMETRY	REYNOLDS NO.	GRID $\theta \times r$	TIME RANGE	TIME STEP SIZES	TOTAL TIME STEPS	NEWTON-KANTOROVICH ITERATIONS EARLY TIME
CYLINDER	15	33 x 40	0–40	.5	80	2
AIRFOIL	1000	65 x 84	0–.2	.01	20	2
AIRFOIL	10000	65 x 84	0–35	.01–.16	260	2
AIRFOIL	100000	129 x 84	0–.1	.01	10	3



$$\frac{\partial U_2}{\partial t} = -\left[\frac{1}{2H}\,\partial\,\frac{(U_1^2 + U_2^2)}{\partial\theta} + U_1\,\omega\right]$$

Fig. 1 Domain of calculation, boundary conditions, and grid notation.

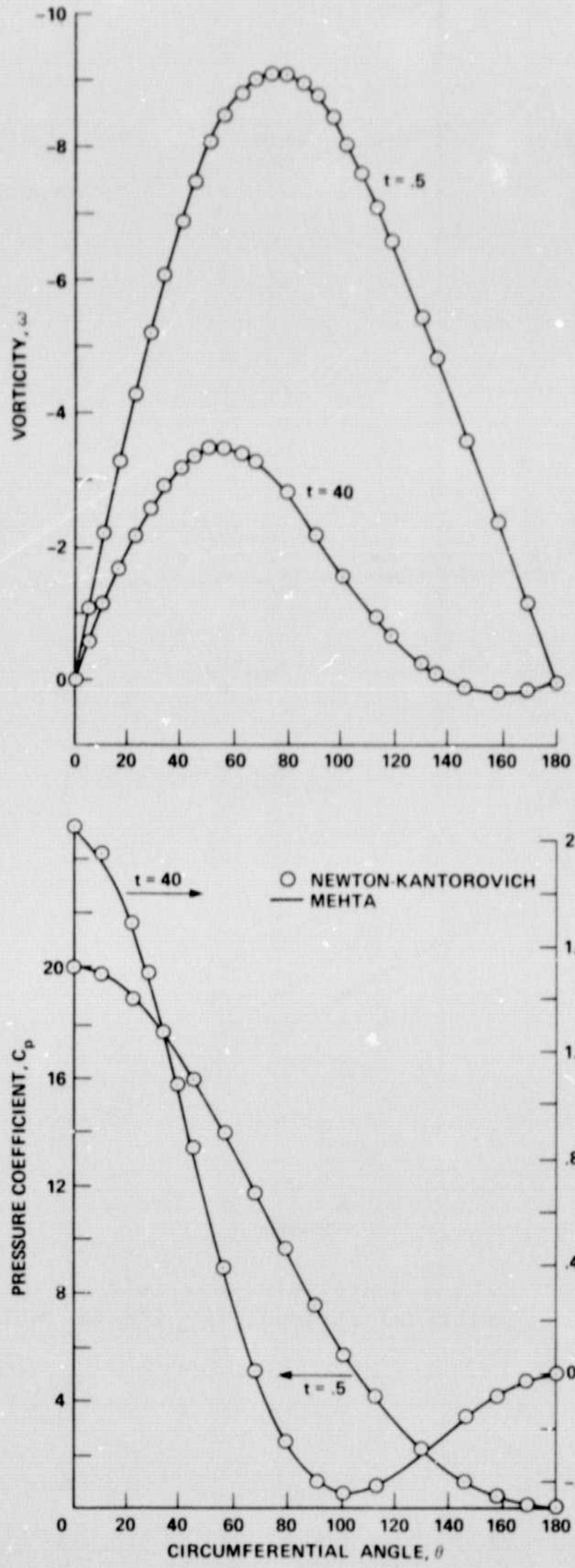


Fig. 2 Surface vorticity and pressure coefficient distributions on a circular cylinder; $Re_\infty = 15$.

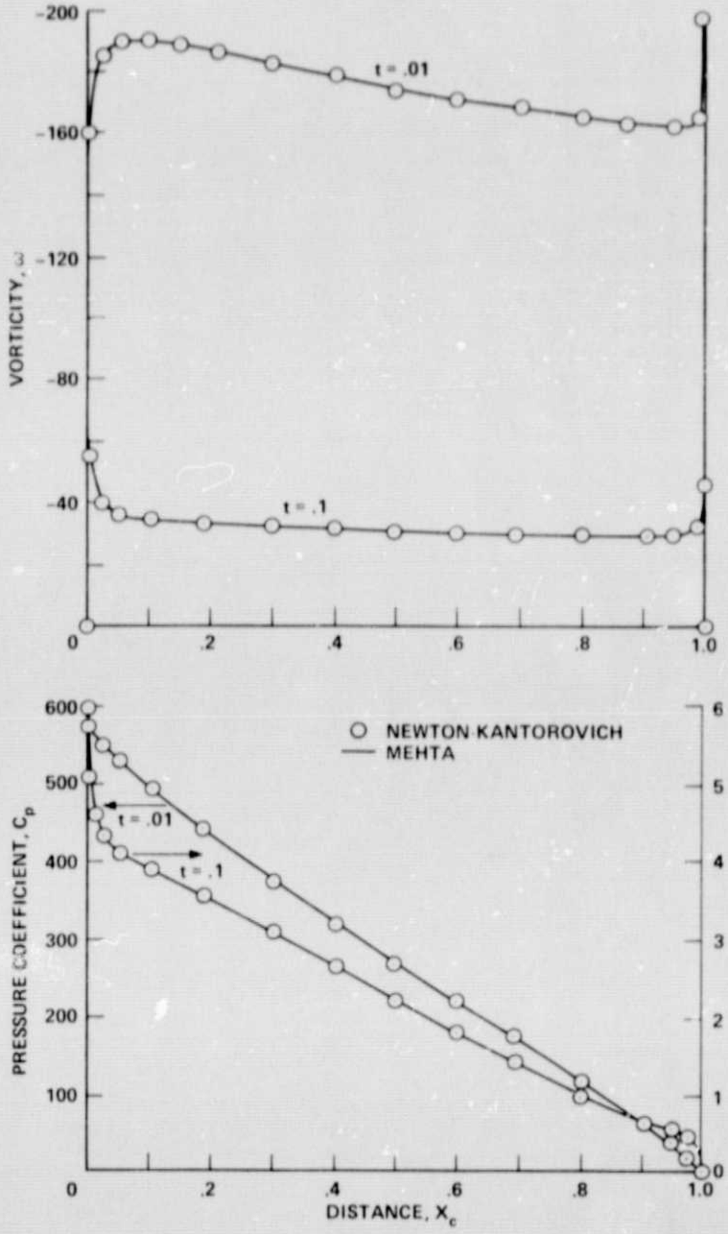


Fig. 3 Surface vorticity and pressure coefficient distributions on a 12% thick symmetrical airfoil; $\alpha = 0°$; $Re_\infty = 1.0 \times 10^3$.

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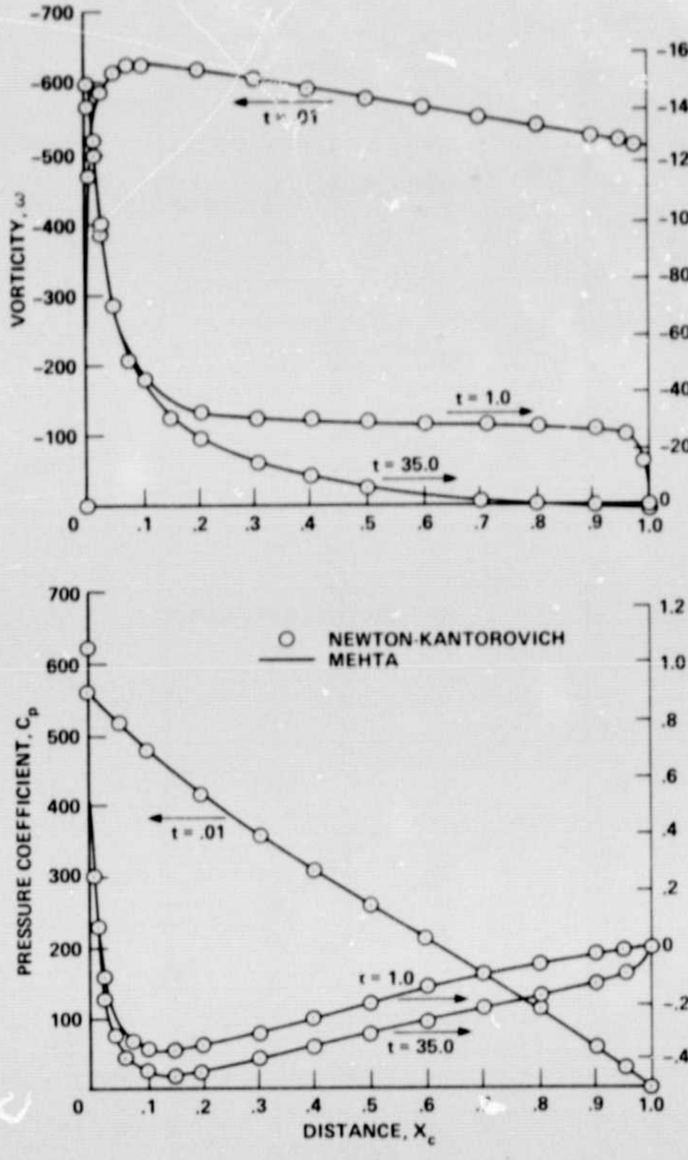

Fig. 4 Surface vorticity and pressure coefficient distributions on a 12% thick symmetrical airfoil; $\alpha = 0°$; $Re_\infty = 1.0 \times 10^4$.

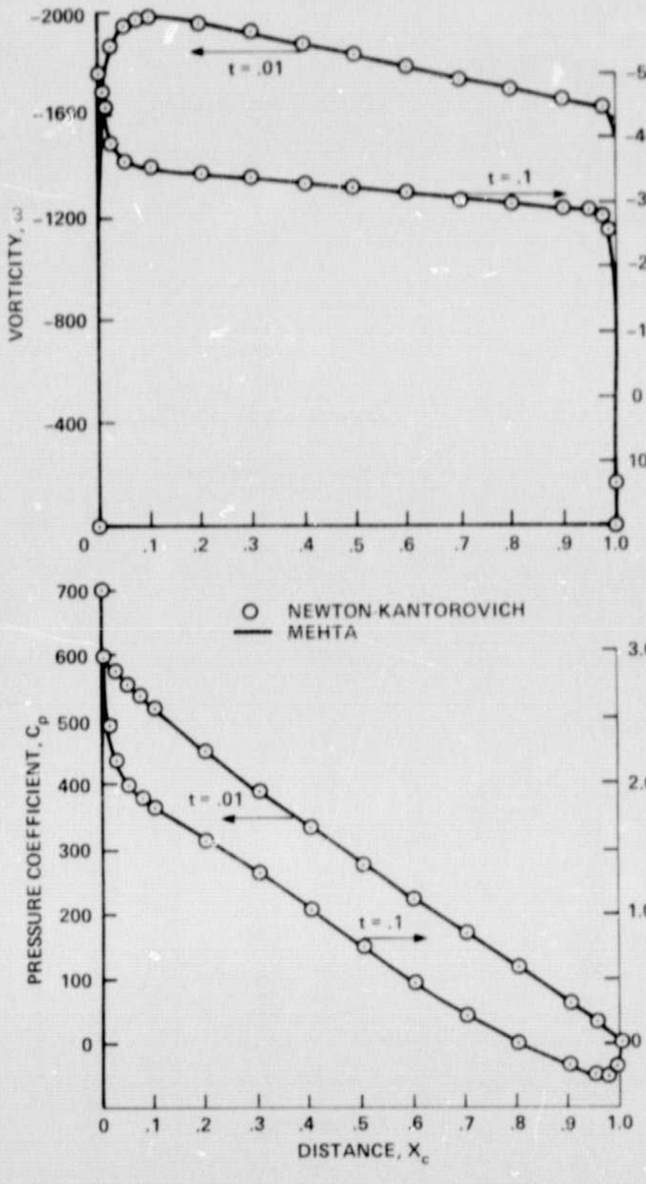

Fig. 5 Surface vorticity and pressure coefficient distributions on a 12% thick symmetrical airfoil; $\alpha = 0°$: $Re_\infty = 1.0 \times 10^5$.

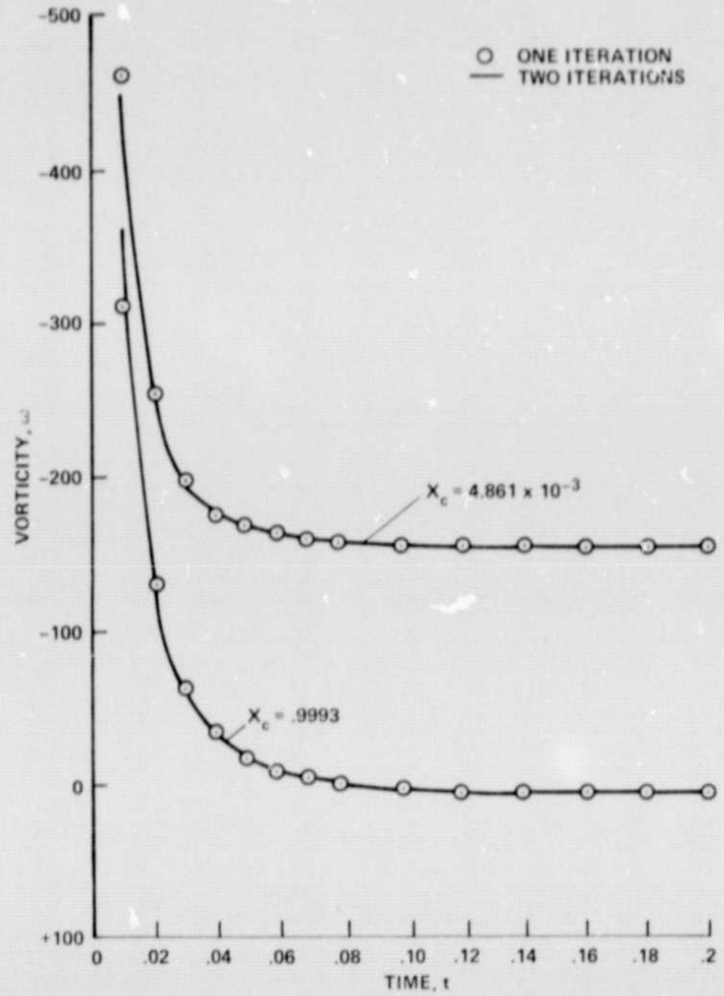

Fig. 6 Vorticity as a function of time at two points on the surface as computed using different number of Newton-Kantorovich iterations for airfoil; $\alpha = 0°$; $Re_\infty = 1.0 \times 10^4$.

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16. Abstract

An implicit finite difference scheme based on the Newton-Kantorovich technique is developed for the numerical solution of the nonsteady, incompressible, two-dimensional Navier-Stokes equations in conservation-law form. The algorithm is second-order-time accurate, noniterative with regard to the nonlinear terms in the vorticity transport equation except at the earliest few time steps, and spatially factored. Numerical results are obtained with the technique for a circular cylinder at Reynolds number 15 as well as a symmetrical airfoil at Reynolds numbers of 10^3, 10^4, and 10^5. The results indicate that the technique is in excellent agreement with other numerical techniques for all geometries and Reynolds numbers investigated, and indicates a potential for significant reduction in computation time over other current iterative techniques.

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